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ASYMPTOTIC MODEL FOR GENERATION OF THREE-DIMENSIONAL VORTEX STRUCTURES IN A BOUNDARY LAYER

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Nonlinear processes of generation of three-dimensional vortex structures play an important role in the distortion of a laminar flow regime in a boundary layer [1-3]. Using the methods of small linearity theory one can determine the instabilities responsible for the formation of three-dimensional structures and describe the initial stage of their evolution (see, for example, [4-8]). Another research trend is concerned with the development of methods for direct numerical solution of the Navier-Stokes equations. Application of few-mode models allows for the construction of three-dimensional structures slightly above critical [9, 10]. The effects of strong nonlinearity are described with the help of complicated numerical schemes which can be realized only using supercomputers [11-15].

Direct numerical simulation does not always provide insight into the mechanisms of the transition processes or an estimate of the degree of their universality. Moreover, it often requires preliminary information obtained from analytical consideration. In this connection, development of physical models that reveal the principal qualitative features of the nonlinear structures acquires significance. Such models can be based on the application of asymptotic methods that go beyond the scope of the weak nonlinearity approximation.

Formation of three-dimensional vortex structures in a nonlinear critical layer in the presence of a wave triplet with fixed wave amplitudes in a flow was studied in [16, 17]. In [18-20] nonlinear models were developed on the basis of the theory of "free interaction," in which the wave number of disturbances is assumed small, and a flow regime with a viscous nonlinear boundary layer is considered. By passing to large disturbance amplitudes (which is, in fact, equivalent to passage to the inviscid limit), in the equations of free interaction, a Benjamin–One equation for two-dimensional nonlinear structures in the boundary layer was obtained in [18, 20]. Its soliton solutions were used in [21, 22] to explain the experimental data. The same equation was derived later for an ideal model of boundary layer flow with a piecewise linear velocity profile [23]. In [24] a generalized Benjamin–One equation was proposed for three-dimensional waves in the boundary layer, which was solved numerically in [25]. However, the equation disregarded the three-dimensional flow instability, which induces the growth of small disturbances.

In the present paper we consider the process of formation of three-dimensional structures as a result of dynamic saturation of the three-dimensional nonlinear instability of small initial disturbances. We consider inviscid plane–parallel flow in a boundary layer with a smooth velocity profile. Dynamic equations are derived using asymptotic expansions in the small wave number.

1. Boundary-value Problem for Long-wave Nonlinear Disturbances in an Ideal Flow. The equations of the principal approximation of the asymptotic theory for long-wave three-dimensional disturbances in a boundary layer obtained in [19] from the free interaction approximation can also be derived reasoning from the equations of an ideal flow. Let us consider briefly the structure of the asymptotic expansions for this case.

We write the Euler equation for ideal incompressible fluid flow in nondimensional form, normalizing to the velocity of the free flow U_∞ and the thickness of the boundary layer δ

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla\mathbf{v}) = -\nabla p, \quad (\nabla \cdot \mathbf{v}) = 0. \tag{1.1}$$

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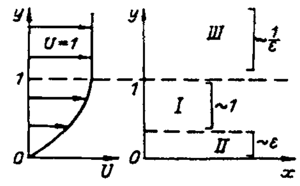

Fig. 1

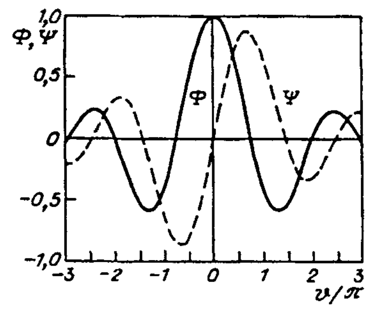

Fig. 2

Here $v = (v_1, v_2, v_3)$ is the velocity field and p is the pressure. We reckon the coordinate x_1 in the direction of the primary flow; x_3, across the flow; and x_2, along the normal to the wall $x_2 = 0$. We introduce a small parameter ε governing the order of magnitude of wave numbers of the disturbance in the x_1 and x_3 directions. We introduce accordingly the variables $x = \varepsilon x_1$, $z = \varepsilon x_3$, and $y \equiv x_2$. To derive the equations of the principal approximation in ε within the framework of the multiscale method, it suffices to use a single time $\tau = \varepsilon^2 t$.

We separate three domains with respect to the coordinate y, as shown in Fig. 1. We use the variable y in the main domain I, the variable $Y = y/\varepsilon$ in the boundary domain II, and the coordinate $S = \varepsilon y$ in the external domain III.

The solution of (1.1) in the domain I can be written

$$\begin{aligned} v_1 &= U(y) + \varepsilon v_1^{(1)} + \varepsilon^2 v_1^{(2)} + \ldots, \qquad & v_2 &= \varepsilon^2 v_2^{(2)} + \varepsilon^3 v_2^{(3)} + \ldots, \\ v_3 &= \varepsilon^2 v_3^{(2)} + \ldots, & p &= \varepsilon^2 p^{(2)} + \varepsilon^3 p^{(3)} + \ldots, \end{aligned} \tag{1.2}$$

where U(y) is the velocity profile of the primary flow ($U = 1$ when $y > 1$, see Fig. 1). The expansions for the domain III are as follows:

$$\begin{aligned} v_1 &= U(y) + \varepsilon^2 \bar{v}_1^{(2)} + \ldots, \qquad & v_2 &= \varepsilon^2 \bar{v}_2^{(2)} + \ldots, \\ v_3 &= \varepsilon^2 \bar{v}_3^{(2)} + \ldots, & p &= \varepsilon^2 \bar{p}^{(2)} + \ldots \end{aligned} \tag{1.3}$$

We construct the solution in the domain II in the form

$$\begin{aligned} v_1 &= \varepsilon V_1^{(1)} + \ldots, \qquad & v_2 &= \varepsilon^3 V_2^{(3)} + \ldots, \\ v_3 &= \varepsilon V_3^{(1)} + \ldots, & p &= \varepsilon^2 P^{(2)} + \ldots \end{aligned} \tag{1.4}$$

The structure of expansions (1.2)-(1.4) corresponds to the class of solutions describing the saturation regime of nonlinear instability. In this class of solutions the characteristic time scale is determined by the frequency of harmonic waves of the linear problem and is of the order of magnitude $1/\varepsilon^2$, while the nonlinear boundary domain II involves the resonance level determined by these waves. Hence follow the above normalization of time t and coordinate y in the domain II.

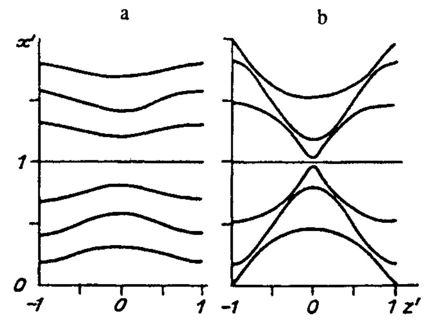

Fig. 3

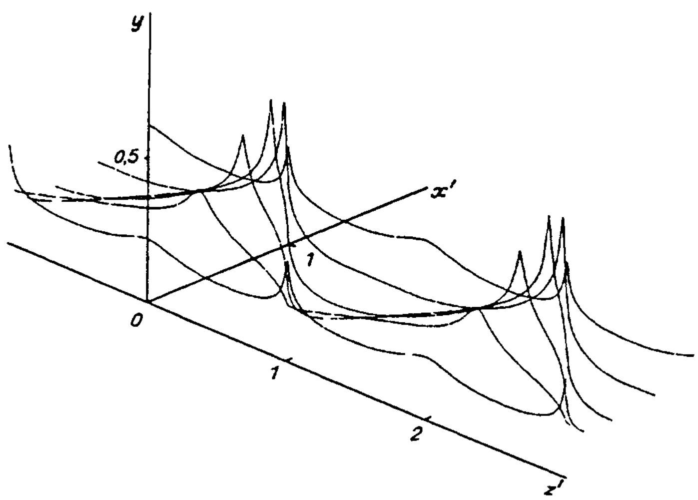

Fig. 4

To simplify the transformations, we introduce the operators $\hat{k} = i\partial/\partial x$ and $\hat{h} = -i\partial/\partial z$, which will be handled as numbers; this rule is understood in the sense of the Fourier transform with respect to x and z. For example, a functional relation of the form $g_1 = F(\hat{k}, \hat{h})g_2$ means that the Fourier transforms

$$\hat{g}_{1,2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_{1,2}\exp(-ikx - ihz)\,dx\,dz$$

obey the relationship $\hat{g}_1 = F(k, h)\hat{g}_2$. Then matching the expansions for v_2 and p in the domains I and III and the expansions for p in the domains I and II, we obtain

$$v_1^{(1)} = A(x,z,\tau)U'(y), \quad v_2^{(2)} = -A_x(x,z,\tau)U(y),$$
$$P^{(2)} = p^{(2)} \equiv P = \frac{\hat{k}^2 A}{\sqrt{\hat{k}^2+\hat{h}^2}}, \quad \tilde{p}^{(2)} = \exp(-S\sqrt{\hat{k}^2+\hat{h}^2})P, \tag{1.5}$$

where $A(x, z, \tau)$ determines the disturbance profile of the longitudinal velocity in the "horizontal" directions x, z; $U'(y) = dU/dy$. The principal term of expansion (1.2) for v_3 is expressed in terms of the pressure: $v_3^{(2)} = -[\hat{h}/\hat{k}U(y)]P$. One can show that the deviation of the material surface from the level $y = y_0$ occupied in the domain I of undisturbed flow, to within terms $\sim\varepsilon$, is

$$y - y_0 = -\varepsilon A. \tag{1.6}$$

The equations for flow in the boundary domain II have the form [19, 20]

$$\begin{gathered}
\frac{\partial V_1^{(1)}}{\partial \tau} + V_1^{(1)}\frac{\partial V_1^{(1)}}{\partial x} + V_2^{(3)}\frac{\partial V_1^{(1)}}{\partial Y} + V_3^{(1)}\frac{\partial V_1^{(1)}}{\partial z} = -\frac{\partial P}{\partial x}, \\
\frac{\partial V_3^{(1)}}{\partial \tau} + V_1^{(1)}\frac{\partial V_3^{(1)}}{\partial x} + V_2^{(3)}\frac{\partial V_3^{(1)}}{\partial Y} + V_3^{(1)}\frac{\partial V_3^{(1)}}{\partial z} = -\frac{\partial P}{\partial z}, \\
\frac{\partial V_1^{(1)}}{\partial x} + \frac{\partial V_2^{(3)}}{\partial Y} + \frac{\partial V_3^{(1)}}{\partial z} = 0.
\end{gathered} \tag{1.7}$$

They are solved with the boundary conditions for $Y \to \infty$, which follow from the conditions of matching of the solutions in the domains I and II in the order ε and the impermeability condition at the wall (see also [19, 20]):

$$V_1^{(1)} \to U_0'(Y + A), \quad V_3^{(1)} \to 0 \quad \text{for} \quad Y \to +\infty; \tag{1.8a}$$

$$V_2^{(3)} = 0 \quad \text{for} \quad Y = 0. \tag{1.8b}$$

Here $U_0' = U'(0)$. Condition (1.8a) for $V_3^{(1)}$ is determined by the behavior of the above function $v_3^{(2)}$ as $y \to 0$. It follows from matching of the expansions for v_3 that $V_3^{(1)}$ decreases as $Y \to \infty$ like $1/Y$. As a result, the longitudinal and transverse velocity components are of the same order with respect to ε in the domain II and of different order in the domain I. The variable $v_2^{(3)}$ can be excluded from (1.7) by integrating the third equation with respect to y taking account of (1.8b). Therefore, Eqs. (1.7) with boundary conditions (1.8b) and relation (1.5) between P and A form a closed boundary-value problem for the domain II. It should be note that the integral representation of the relation between P and A given in [19] can be obtained from the operator representation by means of the formula for the spectral expansion of a spherical wave (see [26, p. 368]).

For two-dimensional disturbances ($\partial/\partial z = 0$), substitution of the asymptotic expansions (1.8a) into (1.7), taking into account the functional relation between P and A, gives the Benjamin–Ono equation for A [20]. In the three-dimensional problem matching of (1.4) and (1.2) taking into account terms of order ε^2 in (1.2) or direct calculation of the asymptotic behavior of the solutions (1.7) as $Y \to \infty$ adds terms $O(1/Y)$ (1.8), which in formal continuation to the wall results in a singularity (the functional representation of these terms is given in [20]). This singularity is due to the resonant character of the interaction between the wave and fluid particles in the domain II, the drift velocity of the particles being of the same order ε as the characteristic wave velocity.

In essence, the domain II is a nonlinear critical wall flow layer, which is generated by a slow wave propagating in the main flow. In the two-dimensional theory the equations of the principal approximation do not involve the profile curvature of the primary flow in the domain II, so that the vorticity remains constant in the domain. In the three-dimensional problem the compression and dilation of vortex tubes enters the picture, generating vorticity disturbances even when the vorticity of the primary flow is constant. Resonance disturbances of the particle trajectories are responsible for the appearance of the disturbance peak of the vorticity components with respect to x and z in the domain II, which achieve values O(1) in the nonlinear regime. The asymptotic expansions of the solution of (1.7) as $Y \to \infty$ no longer coincide with it in the whole domain II, and a single evolution equation can no longer be written for A [20].

The generation of vorticity disturbances in the resonance domain II plays an important role in the flow dynamics, since it produces three-dimensional nonlinear instability. Resonant nonlinear instability in an inviscid flow with a piecewise linear velocity profile was noted in [27]. Analogous solution can be constructed for the problem (1.7) and (1.8) using the expansion in a small disturbance amplitude. Linearizing Eqs. (1.7) with respect to small disturbances of the velocity of the primary flow ($V_1^{(1)} = U_0'Y$) and considering the solutions as three-dimensional waves $\sim \exp(ikx + ihz - ikc\tau)$, one can show that the phase velocity is $c = (1/U_0')\sqrt{k^2 + h^2}$. With $h_1/k_1 = \sqrt{3}$ the condition of resonance of a symmetric pair of oblique waves (k_1, $\pm h_1$) with a two-dimensional wave ($2k_1$, 0) is held. This resonance results in a degenerate nonlinear instability when the amplitude

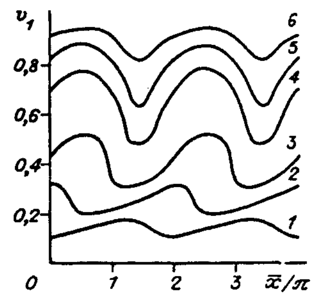

Fig. 5

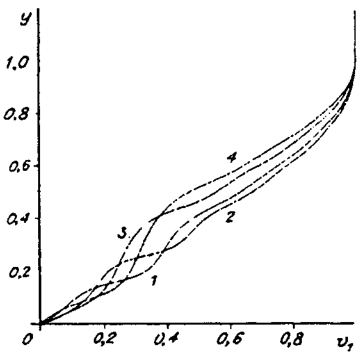

Fig. 6

of the two-dimensional wave is constant, whereas oblique waves grow exponentially [27]. The estimates show that the exponential growth of the oblique waves is not continued up to the saturation stage (probably, there is an interval of their "explosive" growth [27]).

The existence of nonlinear instability in a viscous flow has long been known [4-6]. The similarity of the processes of nonlinear development of the three-dimensional structures in viscous and ideal flows observed in numerical simulation [11, 13] can be attributed to the presence of instability in an inviscid flow. From the viewpoint of a general wave theory this type of instability can be classified as negative nonlinear Landau absorption, since its existence is concerned with the growth of kinetic energy of wave disturbance owing to its nondissipative resonant interaction with the medium. One can assume that nonresonant nonlinear instability, similar to that found numerically in [28] for a viscous flow, is also possible in an inviscid flow. A "rapid" transition toward turbulence in a boundary layer was studied in this paper with the introduction of a disturbance as an arbitrary symmetric pair of oblique harmonics in a flow. The transition was accelerated by eliminating the stage of slow evolution of two-dimensional instability.

2. Lagrangian Coordinates for Resonant Particles. To elucidate qualitative features of the behavior of the disturbances in the resonance domain II, it is convenient to transform to Lagrangian coordinates in (1.7). We define the Lagrangian variables ξ, η, and ζ as the coordinates of fluid particles x, Y, and z at $\tau = 0$. Setting the characteristic disturbance wave numbers along the x and the z axes equal to k_1 and h_1, we introduce the normalized variables:

$$(\bar{x},\bar{\xi}) = k_1(x,\xi), \quad (\bar{z},\bar{\zeta}) = h_1(z,\zeta),$$
$$(\bar{\eta},\bar{Y},\bar{A}) = \frac{U_0'^2}{k_1}(\eta,Y,A), \quad \bar{\tau} = \frac{k_1^2}{U_0'}\tau, \quad \bar{P} = \frac{U_0'^2}{k_1^2}P,$$
$$(u,\bar{c}) = \frac{U_0'}{k_1}(V_1^{(1)},c), \quad w = \frac{h_1 U_0'}{k_1^2}V_3^{(1)}. \tag{2.1}$$

Unless specified otherwise, the bar over the normalized variables is omitted from now on. Then Eqs. (1.7) have the form

$$\frac{dx}{d\tau} = u, \quad \frac{du}{d\tau} = -\frac{\partial P}{\partial x}, \quad \frac{dz}{d\tau} = w, \quad \frac{dw}{d\tau} = -\gamma^2 \frac{\partial P}{\partial z}, \tag{2.2}$$

where $\gamma = h_1/k_1$. The first two equations in the left-hand side of (2.2) should be considered to define the x- and z- components of the trajectories of fluid particles within the given order of accuracy of specification of the velocities. Equations (2.2) show that the motion of the fluid particles in projection onto the x, z plane is autonomous (does not depend explicitly on their vertical displacement) and is similar to the motion of point masses in potential field. The horizontal components of the motion govern the vertical displacement of the particles via the continuity equation, which can be written

$$K_1 \frac{\partial Y}{\partial \xi} + K_2 \frac{\partial Y}{\partial \eta} + K_3 \frac{\partial Y}{\partial \zeta} = 1. \tag{2.3}$$

Here $K_1 = z_\eta x_\zeta - z_\zeta x_\eta$; $K_2 = z_\zeta x_\xi - z_\xi x_\zeta$; $K_3 = z_\xi x_\eta - z_\eta x_\xi$. If instability develops from very small initial disturbances and the case $|A| >> |A(x, z, 0)|$ is considered, the initial conditions corresponding to undisturbed flow can be taken for the particles

$$x(0) = \xi, \quad z(0) = \zeta, \quad u(0) = \eta, \quad w(0) = 0. \tag{2.4}$$

The initial disturbances can be specified in another way by including in (2.2) external action in an explicit form (by introducing an addition to P in (1.5)) in (2.2). Equation (2.3) is solved with the initial condition $Y = 0$ at $\eta = 0$. Conditions (1.8a) take the form $u \to Y + A$, $w \to 0$ as $Y \to \infty$ and actually describe A in terms of the variable P, thus closing the system of equations A together with (1.5).

A system of equations similar to (2.2) and (2.3) was studied in [29, 30] in solving the problem of the evolution of a "flat eddy" in a boundary layer. In [29] separation into domains with respect to y was not carried out, and the pressure was taken into account within the context of perturbation theory as a factor correcting the trajectories in free flight out of the particles. Some general properties of the solutions (2.3) mentioned in [29, 30] will be used below.

The characteristics of the quasilinear equation (2.3) for Y are given by the system of equations [30]

$$\frac{d\xi}{dY} = K_1, \quad \frac{d\eta}{dY} = K_2, \quad \frac{d\zeta}{dY} = K_3. \tag{2.5}$$

This family of characteristics can be parameterized by the values of the Lagrangian coordinates ξ_0 and ζ_0 of the particles on the plane $Y = \eta = 0$. Calculating the increments dx and dz with account of (2.5), one can show that the values of x and z remain invariant along the characteristics. In this case the derivative d/dY in (2.5) coincides with the partial derivative with respect to Y when x and z are fixed [29]. From (2.5) it is obvious that at those points where $K_1 = K_2 = K_3 = 0$ an infinity strong tension of the initial fluid element along Y emerges, and hence a singularity of the ascent of the fluid particles can occur [29].

3. Formation of Three-Dimensional Vortex Structures. In the present paper the solutions of the problem stated in Section 2 are analyzed. The analysis is intended to predict principal tendencies of nonlinear evolution of three-dimensional periodical disturbances at the stage of saturation of nonlinear instability.

Let us assume that small initial disturbance is of the form of a symmetric pair of oblique waves with wave vectors (k_1, $\pm h_1$), and one of the two types of instability mentioned in Section 1 is realized. Then as the instability enters the saturation stage the field A can be represented as (variables are normalized in accordance with (2.1)) $A = 2A_0(\tau)\cos(x - c\tau)\cos z$ ($c = \sqrt{1+\gamma^2}$, A_0 is the amplitude function). In this case the equation for the pressure is

$$P = 2B_0(\tau)\cos(x - c\tau)\cos z, \tag{3.1}$$

where $B_0 = A_0/\sqrt{1+\gamma^2}$. Unidirectional energy exchange between the central flow and the waves, which ceases as the instability is saturated, is characteristic of the stage of weak nonlinearity. According to the normalizations of Sections **1** and **2**, when the flow regime with strong nonlinearity is realized in the domain II, the inverse rise time of the waves is comparable to their frequency c ~ 1 in the linear problem. When the growth rate of the waves approaches this value, the growth of the fundamental harmonics (3.1) terminates and one can take B_0 = const ~1. In qualitative analysis the value const remains indeterminate and is a free parameter of the problem. It should be emphasized that the two-harmonic approximation for pressure (3.1) is used only to determine the trajectories of fluid particles. The total field A and pressure P in this model are found from the condition for matching with the solution in the domain I and emerge as strongly anharmonic functions of x and z at the instability saturation stage (see below). In this case it is assumed that a leading part in the formation of the vortex structure in the domain II is played by the pressure field (3.1) instead of the pressure variation due to the evolution of the structure.

We seek a solution of the system (2.2) and (3.1) with B_0 = const as a perturbation series in the small evolution time τ: $x = \xi + \eta\tau + x^{(2)} + \ldots$, $z = \zeta + z^{(2)} + \ldots$, where $x^{(2)}, z^{(2)} \sim \tau^2$, etc. Applying the initial conditions (2.4), we obtain to within terms $\sim\tau^2$ the expressions

$$
\begin{aligned}
x &= \xi + \eta\tau + B_0\tau^2\cos\zeta[\Phi(\vartheta)\sin(\xi+\vartheta) - \Psi(\vartheta)\cos(\xi+\vartheta)],\\
z &= \zeta + \gamma^2 B_0\tau^2\sin\zeta[\Psi(\vartheta)\sin(\xi+\vartheta) + \Phi(\vartheta)\cos(\xi+\vartheta)],
\end{aligned}
\tag{3.2}
$$

where $\vartheta = (\eta - c)\tau$; $\Phi = 2(\cos\vartheta - 1 + \vartheta\sin\vartheta)/\vartheta^2$; $\Psi = 2(\sin\vartheta - \vartheta\cos\vartheta)/\vartheta^2$. Functions Φ and Ψ are shown in Fig. 2.

Let us use the analytic representation (3.2) to find the right-hand sides of (2.5). It should be emphasized that the formation of a nonlinear disturbance is determined by resonant particles (in the vicinity of the principal maximum of the function $\Phi(\eta)$ in Fig. 2), the trajectories of the particles are the most affected by the wave field.

With the above given initial disturbances, the projection of the particle trajectories onto the (x, z) plane enjoy rigorous properties of periodicity and symmetry (like in [17]). Mirror symmetry with respect to the planes $z = \pi n$ (n is an integer) as well as the symmetry of the checkerboard type takes place and are manifested in the invariance of the pattern of trajectories under simultaneous translation along x and z by the half-period π. The planes $z = \pi n$ are material surfaces at which the functions $K_{1,2,3}$ take the form

$$
K_1 = -z_\zeta x_\eta, \quad K_2 = z_\zeta x_\xi, \quad K_3 = 0. \tag{3.3}
$$

It follows from (3.3) that $K_{1,2,3} = 0$ when $\partial z/\partial\zeta = 0$. Within the framework of approximation (3.2) this condition holds for the first time on the resonance surface $\eta = c$ at the points of intersection of the material lines $\xi = (2n + 1)\pi$ and $\xi = 2n\pi$ with the symmetry planes $\zeta = 2m\pi$ and $\zeta = (2m + 1)\pi$, respectively (m and n are integers). This occurs at the time $\tau = \tau_\infty \equiv 1/\gamma\sqrt{B_0}$. It should be noted that the projections of the material lines $\xi = n\pi$, $\eta = c$ onto the plane x, z remain straight at all times. As was noted in Section 2, at the time $\tau = \tau_\infty$ a singular ascent of fluid particles along Y arises at the points mentioned. Taking into account the behavior of the functions Φ and Ψ in Fig. 2, we find the half-width Δ_c of the resonance layer of particles with respect to η from the condition $\vartheta \approx \Delta_c\tau_\infty = \pi$. We use Δ_c below as a free parameter, expressing the wave amplitude in terms of it: $B_0 = (\Delta_c/\pi\gamma)^2$. For $\Delta_c = c$ the critical layer formed by the pressure field (3.1) encompasses the whole domain from the resonance level Y = c to the wall as the instability enters the saturation stage.

Figure 3 shows instantaneous (time-freeze) plots of the projections of the material curves $\bar\eta = \bar c$, $\xi = 0.2(n - 1)\pi$ (n = 1, ..., 11) onto the plane $x' = (\bar x - \bar c\bar\tau)/\pi$, $z' = \bar z/\pi$ for $\gamma = 1$, $\Delta_c = 0.4\bar c$, and $\bar\tau = 0.5\tau_\infty$ (a); $\bar\tau = \tau_\infty$ (b) (here we reinstate the bar over normalized variables (2.1)). In this statement of the problem the material lines clearly coincide with the vortex lines of the flow. The line patterns in Fig. 3 should be continued periodically by successive x' and z' translations with period 2. It is evident that as $\tau \to \tau_\infty$, the vortex lines acquire lambda-shaped kinks with tips focused at the points of singular ascent of the fluid particles.

To justify the above approach, we have plotted instantaneous projections of the material lines based on the numerical solution of Eqs. (2.2) and (3.1) with B_0 = const. We obtained projections close to those shown in Fig. 3 with an insignificant (less than 10%) difference in the formation time of the singularity. Hence it follows that the formulas (3.2) derived by the perturbation method for small τ provide a qualitatively correct description of the behavior of complete solution of Eqs. (2.2) with the pressure (3.1) until the derivative z_ζ vanishes.

We now show that the disturbances of the particle projections in approximation (3.2) can be matched with the trajectory disturbances emerging in the growth stage of the waves. To find the disturbances of the trajectories in the small nonlinearity stage, we set $B_0 = \mu \exp(\nu\tau)$ in (3.1) ($\mu << 1$ is a small parameter, and $\nu > 0$ is the local growth rate of the wave) and construct a solution of (2.2) using perturbation theory for μ. Assuming ν = const, to within terms of the order μ we have

$$
\begin{aligned}
x &= \xi + \eta\tau + 2B_0\nu^{-2}\cos\zeta[\tilde{\Phi}(\rho)\sin(\xi+\vartheta) - \tilde{\Psi}(\rho)\cos(\xi+\vartheta)],\\
z &= \zeta + 2B_0\gamma^2\nu^{-2}\sin\zeta[\tilde{\Psi}(\rho)\sin(\xi+\vartheta) + \tilde{\Phi}(\rho)\cos(\xi+\vartheta)],
\end{aligned}
\tag{3.4}
$$

where $\rho = (\eta - c)/\nu$; $\tilde{\Phi} = (1 - \rho^2)/(1 + \rho^2)^2$; $\tilde{\Psi} = 2\rho/(1 + \rho^2)^2$. It should be noted that the pairs of functions Φ, $\tilde{\Phi}$ and Ψ, $\tilde{\Psi}$ are structurally similar in the vicinity of resonance. Equating $B_0(\tau)$ in (3.4) to the value B_0 = const in (3.2) and matching the disturbance amplitudes of the particle trajectories in (3.2) and (3.4), we obtain $\nu = \sqrt{2}/\tau$ at the resonance level $\eta = c$. In this case the resonance width with respect to η, determined by the principal maxima of the functions Φ and $\tilde{\Phi}$, is approximately the same. It is natural to take the matching moment τ comparable to the value τ_∞, but not too close to it. For example, with $\tau = (1/2)\tau_\infty$ we have $\nu = 2\gamma\sqrt{2B_0} \sim 1$, as it must when the instability passes into the saturation stage. Thus, formulas (3.2) account for qualitative behavior features of horizontal projections of the flow vortex lines when the instability of initially small disturbances is saturated.

Before describing the results for the three-dimensional flow of calculations we write the combined solution of the problem in the domains I and II. For the total longitudinal velocity the solution can be written (reinstating the bar over the normalized variables)

$$
v_1 = U(y) + \frac{\varepsilon k_1}{U_0'^2}U'(y)[\overline{u}(\overline{x},\overline{Y},\overline{z},\overline{\tau}) - \overline{Y}].
\tag{3.5}
$$

It can be shown that expression (1.6) for the deviation of material surface in the domain I coincides with the asymptotic value in the domain II as $Y_0 = y_0/\varepsilon \to \infty$. Therefore, the combined solution for the total coordinate y of points of the material surface takes the form

$$
y = \frac{\varepsilon k_1}{U_0'^2}\overline{Y}(\overline{x},\overline{Y}_0,\overline{z},\overline{\tau}),
\tag{3.6}
$$

where $\overline{Y}_0 = (\overline{Y}/Y)Y_0$ (see (2.1)). When $\overline{Y}_0 = \overline{\eta}$ material (vortex), lines lying in the plane $\overline{\eta}$ = const are also on the surface (3.6). Therefore, formula (3.6) can be used to calculate the lift of fluid particles at these lines.

Where spindle-shaped dips occur in the oscillograms of longitudinal velocity, values of the wave number (related to $1/\delta$) $\varepsilon k_1 \sim 0.7$ and the phase velocity of the waves (related to U_∞) $\varepsilon c \approx 0.38$ have occurred in experiments [2]. In this case the critical level is at the height $y \approx 0.24$ (it should be noted that $U_0' \approx 1.6$ for the Blasius profile). In the formal application of the asymptotic solution with this phase velocity a significant part of the boundary layer belongs to the domain II. In the context of the asymptotic theory, according to Section 1, $\varepsilon c = (\varepsilon k_1/U_0')\sqrt{1+\gamma^2}$. Presented below are the results of calculations carried out for $\gamma = 1$, $\varepsilon c = 0.3$ ($\varepsilon k_1 = 0.34$) and $\Delta_c = 0.4c$. The velocity profile is approximated in (3.3) by a function close to the Blasius profile: $U = 0.4y^6 - y^4 + 1.6y$ for $0 \le y \le 1$, and $U = 1$ for $y \ge 1$. It should be noted that analogous results are obtained for other values of the parameters and with neglect of the dispersion relation of the phase velocity to the wave number.

The calculation results for several principal characteristics of the flow field are presented in Figs. 4-6. Figure 4 shows an instantaneous plot of the system of three-dimensional vortex lines existing at the initial instant at the resonance level $\overline{\eta} = \overline{c}(\overline{\xi} = 0, 2\pi(n - 1)$, $n = 1, \ldots, 6$, x' and z' are the same as in Fig. 3). The graph is plotted at $\tau = 0.95_{\tau\infty}$ within two periods in $\overline{\zeta}$. In the calculations the initial particle coordinates $\overline{\xi}$, $\overline{\zeta}$, and $\overline{\eta}$ were taken as initial values, and their coordinates $\overline{x}$, $\overline{z}$ were calculated for the time $\overline{\tau}$ from formulas (3.2). The initial particle coordinates $\overline{\xi}_0$, $\overline{\zeta}_0$ (which have the same coordinates $\overline{x}$, $\overline{z}$ at the moment $\overline{\tau}$) in the plane $\overline{\eta} = 0$ were found by solving (3.2) by iteration methods. The unknown value of $\overline{Y}$ was determined by integrating (2.5) numerically from $\overline{Y} = 0$ to the $\overline{Y}$ at which the initial value $\overline{\eta}$ was achieved.

Figure 5 shows the dependences of the total longitudinal velocity v_1 on $\bar{x}$ at different levels with respect to y in the symmetry plane $\bar{z} = 0$ at $\bar{\tau} = 0.97\tau_\infty$ (curves 1-6 correspond to y = 0.1; 0.19; 0.4; 0.6; 0.7; 0.8). Figure 6 shows a sequence of local profiles of the longitudinal velocity at $\bar{\tau} = 0.94\tau_\infty$ (curves 1-4 correspond to $\bar{x} = 0.5(n - 1)\pi$, n = 1-4). In plotting the curves in Figs. 5 and 6, the system (2.5) was integrated with allowance for expressions (3.3) for predetermined values of $\bar{x}$.

The qualitative features of the three-dimensional flow presented in Figs. 4-6 are due to the onset of domains with large vertical elongation of the material elements because of their transverse compression during autonomous motion of the particle system in the given pressure field (3.1). This effect results in local ejections of low-velocity fluid in upper layers of the flow, which are clearly seen in Fig. 4. A pattern of material lines similar to that shown in Fig. 4 was obtained in [15] from direct numerical simulation of the transition. As a result of the ascent of the low-velocity fluid and the increase in its drift velocity by the primary flow at high levels along the y axis, spindle-shaped dips are formed in the longitudinal velocity oscillations along the x axis (Fig. 5). These processes are responsible for the formation of instantaneous longitudinal velocity profiles with inflection points (Fig. 6).

To compare the results with measured data for the time dependence of the longitudinal velocity at a fixed point, one can use the convection hypothesis, according to which x_1 is substituted for $-U_c t$ (where U_c is the disturbance drift velocity). Then one can see that the curves in Figs. 5 and 6 are in good qualitative agreement with the experimental data (see [Figs. 3h, and 11a-d in 2]). We can thus trace the transition from oscillations with overcrested fronts at the level of the critical layer (y $\approx$ 0.19 in Fig. 5) to the spindle-shaped dips at higher levels. The amplitude of the dips can make up a significant part of the freestream velocity. The occurrence of oscillograms with velocity reversal at y > 1 is probably due to the formation of a thin vortex core during the elongation of vortex tubes in the domains of local ascent of the particles (this tendency is observed in Fig. 4). The vortex core amplifies the velocity defect at the dips at high levels along y, but its description is beyond the limits of the flat-eddy approximation.

In keeping with the matching conditions derived in Section 2, we have $\bar{u} - \bar{Y} \to \bar{A}$ as $\bar{Y} \to \infty$. The calculations show that the difference $\bar{u} - \bar{Y}$ actually tends to a fixed function of x. As is clear from Fig. 5, the field $\bar{A}$ and, therefore, the normalized pressure $\bar{P}$ become nonsinusoidal functions of $\bar{x}$ (attenuation of the oscillations as y increases is due to the action of the factor U' in formula (3.5)). This suggests the onset of self-consistent pressure, which probably prevents the formation of a singularity of the particles ascent. The latter is indirectly justified by the results of model calculations described in [29, 30] where the flat-eddy singularity was eliminated with allowance for the pressure gradients induced by the flat-eddy. In this case longitudinal velocity oscillations set in, which in reference to our model can be considered as an analog of multiplication of the number of dips (which is also observed in the transition experiments [2]). It should be added that the patterns of projections of the vortex lines in Fig. 3 are in qualitative agreement with the results of direct numerical simulation of the evolution of a three-dimensional disturbance in an ideal flow [11].

Thus, the asymptotic model for the generation of three-dimensional vortex structures at saturation of nonlinear instability in the boundary layer describes a number of qualitative features of the flow field observed in the final stage of transition in laboratory experiments and numerical simulation. In conclusion we call attention to the principal distinction between the processes of formation of three-dimensional and two-dimensional structures in an ideal flow. Three-dimensional structures are formed as a result of the saturation of nonlinear instability, while the two-dimensional problem for nonlinear disturbances in the absence of viscosity is conservative. Application of the model described in Section 3 to the two-dimensional problem results in overcresting of the slopes of an originally sinusoidal disturbance, consistent with the solution of the Benjamin–Ono equation for the function A with elimination of the pressure term (the term containing the Hilbert transform) from it. In the two-dimensional problem without regard for self-consistent pressure, a singularity of the field gradient A arises, whereas in the three-dimensional problem, the singularity is in the field itself. Self-consistent pressure essentially influences the shape of two-dimensional nonlinear waves, while for three-dimensional disturbances, which in the absence of self-consistent pressure have a symmetric solitonlike form, its role is not likely to be crucial in this sense. Therefore, the model in question fails to provide good results when a sufficiently strong two-dimensional wave is included in the pressure expression (3.1). The generation of three-dimensional structures without growth of the two-dimensional wave, which is characteristic of the inviscid problem, is also possible for a viscous flow exhibiting a rapid transition to turbulence.

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REFERENCES

1. V. V. Kozlov, V. Ya. Levchenko, and V. S. Starik, "Formation of three-dimensional structures in transition to turbulence in a boundary layer," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 6 (1984).
2. Yu. S. Kachanov, V. V. Kozlov, V. Ya. Levchenko, and M. P. Ramazanov, "Nature of K-destruction of a laminar boundary layer," Izv. Sib. Otd. Akad. Nauk SSSR, Ser. Tekh. Nauk, No. 2 (1989).
3. Yu. S. Kachanov, "The mechanisms of formation and breakdown of solitonlike coherent structures in boundary layers," in: Advances in Turbulence 3, A. V. Johansson and P. H. Alfredsson (eds.), Springer-Verlag, Berlin (1991), pp. 42-51.
4. A. D. D. Craik, "Nonlinear resonant instability in boundary layers," J. Fluid Mech., **50**, Part 2 (1971).
5. J. R. Usher, A. D. D. Craik, and F. Hendriks, "Nonlinear wave interactions in shear flows. Part 2. Third-order theory," J. Fluid Mech., **70**, No. 3 (1975).
6. A. G. Volodin and M. B. Zel'man, "Three-wave resonant interaction of disturbances in a boundary layer," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 5 (1978).
7. B. J. Bayly, S. A. Orszag, and T. Herbert, "Instability mechanisms in shear-flow transition," Ann. Rev. Fluid Mech., **20** (1988), pp. 359-391.
8. A. D. D. Craik, Wave Interaction and Fluid Flows, Cambridge (1985).
9. M. A. Gol'dshtik, A. M. Lifshitz, and V. N. Shtern, "Reynolds number for transition in a planar channel," Dokl. Akad. Nauk SSSR, **273**, No. 1 (1983).
10. R. D. Rakhmatullaev and V. N. Shtern, "Threshold of the onset of three-dimensional structures in flat channel flow," Preprint Inst. Teplofiziki, Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1989).
11. A. Leonard, "Vortex simulation of three-dimensional spotlike disturbances in a laminar boundary layer," in: Turbulent Shear Flows, Vol. 2, Springer-Verlag, Berlin (1980), pp. 67-77.
12. S. A. Orszag, "Numerical studies of transition in planar shear flows," in: Laminar – Turbulent Transition, Proc. Symp. IUTAM, Springer-Verlag, Berlin (1980), pp. 153-162.
13. E. Laurien and L. Kleiser, "Numerical simulation of boundary layer transition and transition control," J. Fluid Mech., **199** (1989).
14. H. F. Fasel, U. Rist, and U. Konzelmann, "Numerical investigation of three-dimensional evolution in boundary layer transition," AIAAJ., **28**, No. 1 (1990).
15. H. L. Reed, "Gallery of fluid motion," Phys. Fluids A, **4**, No. 9 (1992).
16. V. P. Reutov, "The critical layer and nonlinear waves in wall flows," in: Laminar – Turbulent Transition: Proc. Symp. IUTAM, Springer-Verlag, Berlin (1985), pp. 81-86.
17. V. P. Reutov, "Nonlinear critical layer and formation of linear vortices with interaction of waves in shear flows," Prikl. Mekh. Tekh. Fiz., No. 5 (1987).
18. V. I. Zhuk and O. S. Ryzhov, "Local inviscid disturbances in a boundary layer," Dokl. Akad. Nauk SSSR, **263**, No. 1 (1982).
19. V. I. Zhuk and O. S. Ryzhov, "Three-dimensional inviscid disturbances inducing an intrinsic pressure gradient in a boundary layer," Dokl. Akad. Nauk SSSR, **301**, No. 1 (1989).
20. O. S. Ryzhov, "Formation of ordered vortex structures from unsteady oscillations in a boundary layer," Zh. Vych. Mat. Mat. Fiz., **30**, No. 12 (1990).
21. Yu. S. Kachanov and O. S. Ryzhov, "Formation of solitons in a transition boundary layer. Theory and experiment," Izv. Sib. Otd. Akad. Nauk SSSR, Ser. Tekh. Nauk, No. 1 (1992).
22. Yu. S. Kachanov and O. S. Ryzhov, "Formation of solitons in a transitional boundary layer. Theory and experiment," in: Proc. Boundary Layer Transition Control Conference, Cambridge, U.K. (1991).
23. P. M. Balagondar, S. A. Maslow, and S. Melkonian, "Propagation of finite-amplitude waves in a model boundary layer," Stud. Appl. Math., **76**, No. 2 (1987).
24. V. I. Shrira, "On 'subsurface' waves of the upper quasi-homogeneous ocean layer," Dokl. Akad. Nauk SSSR, **308**, No. 3 (1989).
25. L. Ya. Abramyan, Yu. A. Stepanyantz, and V. I. Shrira, "Inhomogeneous solitons in shear flows of the boundary layer type," Dokl. Akad. Nauk SSSR, **327**, No. 4 (1992).
26. M. B. Vinogradova, O. V. Rudenko, and A. P. Sukhorukov, Wave Theory [in Russian], Nauka, Moscow (1979).

27. V. P. Reutov, "A note on the mechanism of the resonant nonlinear instability in boundary layers," in: Research Reports in Physics. Nonlinear Waves 3, A. V. Gaponov-Grekhov et al. (eds.), Springer-Verlag, Berlin (1990), pp. 129-132.
28. P. J. Schmid and D. S. Henningson, "A new mechanism for rapid transition involving a pair of oblique waves," Phys. Fluids A, **4**, No. 9 (1992).
29. J. R. Russell and M. T. Landahl, "The evolution of a flat-eddy near a wall in an inviscid shear flow," Phys. Fluids, **27**, No. 3 (1984).
30. M. T. Landahl, "Flat-eddy model for coherent structures in boundary layer turbulence," Theoretical Approaches to Turbulence, New York (1985), pp. 137-153.